# Benchmark of adjustable size and coupling to test ODEs solvers

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# Outline

- Introduction and starting point
- Quality criterias of a benchmark for ODE solvers
- Description of the developped benchmark
- Results for two different parallelization strategies
- Conclusions

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# Introduction

- Provide a benchmark of adjustable size and more or less coupled in order to test an ODE solver
- Show the ability of this benchmark to compare two different parallelization strategies for ODE solvers

# Starting point

Starting point of the desired benchmark in a mathematical frame :

$$\begin{cases} t \in I = [t_{start} = 0, t_{end}] \subset \mathbb{R} \\ f : I \times \mathbb{R}^n \to \mathbb{R}^n \\ x \in \mathbb{R}^n \\ \dot{x} = f(t, x) \\ x(t = t_{deb}) = x_0 \end{cases} \quad (1)$$

Main hypothesis : only linear models are considered :

$$\begin{cases} t \in I = [t_{start} = 0, t_{end}] \subset \mathbb{R} \\ B \in \mathbb{R}^n, A \in M_n(\mathbb{R}) \\ x \in \mathbb{R}^n \\ \dot{x} = Ax + B \\ x(t = t_{start}) = x_0 \end{cases} \quad (2)$$

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  - Need of the analytical solution
  - Need to adjust the density of the matrice A
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# Need of the analytical solution

- Better than using Euler scheme with a very small time step
- Validate the solution given by the solver
- Compare the precision of the solutions between two solvers

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# Need to adjust the density of the matrice A

- Initially A is described in the canonical basis $E_0 = (e_1, \ldots, e_n)$ of the state variables of the model.
- Build the base $E_\alpha = (e_{\alpha,1}, \ldots, e_{\alpha,n})$

$$\begin{cases} e_{\alpha,1} &= \cos(\alpha)e_1 + \sin(\alpha)e_2 \\ e_{\alpha,2} &= -\sin(\alpha)e_1 + \cos(\alpha)e_2 \\ e_{\alpha,3} &= e_3 \\ \vdots & \\ e_{\alpha,n} &= e_n \end{cases} \tag{3}$$

- $P_1^{E_0 \to E_\alpha}$ the change of basis matrix.

# Need to adjust the density of the matrice A

- $A_\alpha$ the matrix of the endomorphism of $f$ from $E_\alpha$ into $E_\alpha$

$$A_\alpha = (P_1^{E_0 \to E_\alpha})^{-1} A \, P_1^{E_0 \to E_\alpha} \; = (P_1^{E_0 \to E_\alpha})^t A \, P_1^{E_0 \to E_\alpha}$$

- the new ODE system :

$$(P_1^{E_0 \to E_\alpha})^t \dot{X}(t) = (P_1^{E_0 \to E_\alpha})^t (A \, X(t) + B)$$
$$= (P_1^{E_0 \to E_\alpha})^t A \, P_1^{E_0 \to E_\alpha} (P_1^{E_0 \to E_\alpha})^t \, X(t) + (P_1^{E_0 \to E_\alpha})^t B$$

- finally :

$$\dot{X}_\alpha(t) = A_\alpha X_\alpha(t) + B_\alpha \qquad (4)$$

↪ $A_\alpha$ is more dense than $A$

↪ repeat the process by shifting the rotations until A is dense.

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# 1D heat diffusion model

Fourier law

- metal stick of length L supposed to be thin enough to allow 1D modelling
- temperature maintained constant at both ends
- $\alpha$ constant

The following PDE is :

$$\begin{cases} \dfrac{\partial T}{\partial t} = \dfrac{\partial}{\partial X}\left(\alpha \dfrac{\partial T}{\partial X}\right) \\ T(X, t=0) = T_0(X) \quad \forall X \in [0, L] \\ T(X=0, t) = T_l \quad \forall t \geq 0 \\ T(X=L, t) = T_r \quad \forall t \geq 0 \end{cases} \quad (5)$$

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# Quality criterias of the model

- Analytical solution of (5) is known
- By discretizing the spatial derivative (M.O.L.) :

$$\frac{\partial T(i)}{\partial t} = \frac{\alpha}{dx^2}(T(i+1) - 2T(i) + T(i-1)) \tag{6}$$

  - The size of the model is adjusted
  - The matrices of the problem are given

$$\dot{T}(t) = AT(t) + B \tag{7}$$

with $A = \frac{\alpha}{dx^2}\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & -2 & 1 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & -2 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}$

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# Model complexification : example with five nodes

Five nodes in the stick are considered :

$$\dot{X}(t) = \frac{\lambda}{dx^2} \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} X(t) + \frac{\lambda}{dx^2} \begin{pmatrix} T_l \\ 0 \\ \vdots \\ 0 \\ T_r \end{pmatrix} \tag{8}$$

After 0 rotation :

$$A_0 = \frac{\lambda}{dx^2}\begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

After 1 rotation :

$$A_1 = \frac{\lambda}{dx^2}\begin{pmatrix} -1.13 & -0.50 & 0.866 & 0 & 0 \\ -0.50 & -2.86 & 0.50 & 0 & 0 \\ 0.866 & 0.50 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

After 2 rotations :

$$A_2 = \frac{\lambda}{dx^2}\begin{pmatrix} -1.13 & 0.50 & 0.87 & 0 & 0 \\ 0.50 & -1.78 & 0.12 & 0.87 & 0 \\ 0.87 & 0.12 & -3.08 & 0.50 & 0 \\ 0 & 0.87 & 0.50 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

After 3 rotations :

$$A_3 = \frac{\lambda}{dx^2}\begin{pmatrix} -1.13 & 0.50 & 0.43 & -0.75 & 0 \\ 0.50 & -1.78 & 0.81 & 0.32 & 0 \\ 0.43 & 0.81 & -1.84 & 0.22 & 0.87 \\ -0.75 & 0.32 & 0.22 & -3.25 & 0.5 \\ 0 & 0 & 0.87 & 0.5 & -2 \end{pmatrix}$$

After 4 rotations :

$$A_4 = \frac{\lambda}{dx^2}\begin{pmatrix} -1.13 & 0.50 & 0.43 & -0.37 & 0.65 \\ 0.50 & -1.78 & 0.81 & 0.16 & -0.28 \\ 0.43 & 0.81 & -1.84 & 0.86 & 0.24 \\ -0.37 & 0.16 & 0.86 & -1.88 & 0.29 \\ 0.65 & -0.28 & 0.24 & 0.29 & -3.37 \end{pmatrix}$$

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# Parallelization strategy 1 ("block parallelization")

Let consider a 6x6 matrix with two rotations done :

$$A = \frac{\lambda}{dx^2}\begin{pmatrix} -1.13 & 0.50 & 0.86 & 0 & 0 & 0 \\ 0.50 & -1.78 & 0.13 & 0.86 & 0 & 0 \\ 0.86 & 0.13 & -3.08 & 0.50 & 0 & 0 \\ 0 & 0.86 & 0.50 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

core 1

core 2

## In core 1 :

$$A_{core1} = \frac{\lambda}{dx^2}\begin{pmatrix} -1.13 & 0.50 & 0.86 & 0 & 0 & 0 \\ 0.50 & -1.78 & 0.13 & 0.86 & 0 & 0 \\ 0.86 & 0.13 & -3.08 & 0.50 & 0 & 0 \end{pmatrix}$$

$\rightarrow$ 11 non zero elements

## In core 2 :

$$A_{core2} = \frac{\lambda}{dx^2}\begin{pmatrix} 0 & 0.86 & 0.50 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$\rightarrow$ 9 non zero elements

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# Parallelization strategy 2 ("line parallelization")

Let consider the same matrix

$$A = \frac{\lambda}{dx^2}\begin{pmatrix} -1.13 & 0.50 & 0.86 & 0 & 0 & 0 \\ 0.50 & -1.78 & 0.13 & 0.86 & 0 & 0 \\ 0.86 & 0.13 & -3.08 & 0.50 & 0 & 0 \\ 0 & 0.86 & 0.50 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

core 1
core 2
core 1
core 2
core 1
core 2

**In core 1 :**

$$A_{core1} = \frac{\lambda}{dx^2}\begin{pmatrix} -1.13 & 0.50 & 0.86 & 0 & 0 & 0 \\ 0.86 & 0.13 & -3.08 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

$\rightarrow$ 10 non zero elements

**In core 2 :**

$$A_{core2} = \frac{\lambda}{dx^2}\begin{pmatrix} 0.50 & -1.78 & 0.13 & 0.86 & 0 & 0 \\ 0 & 0.86 & 0.50 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

$\rightarrow$ 10 non zero elements

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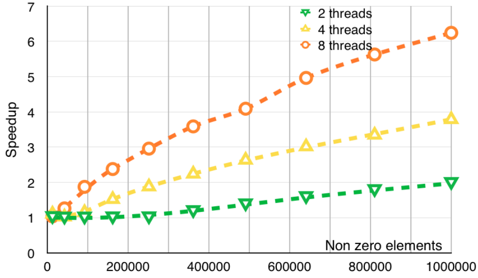
# Model with 1000 nodes

- Heat diffusion with 1000 nodes
- Every 100 rotations the computation time for both solvers is stored
- Number of non zero elements: 3000 $\rightarrow$ 1000000

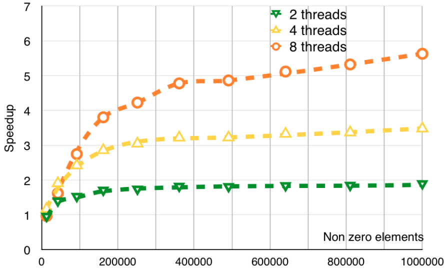
# Results for the two different parallelization strategies

Speedup : computation time with one core / computation time with n cores

Block parallelization




Line parallelization



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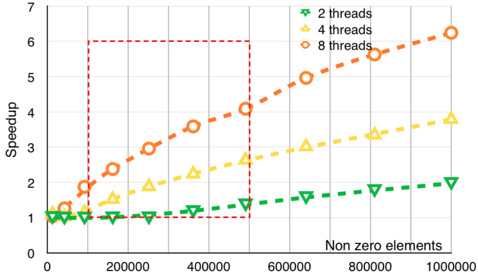
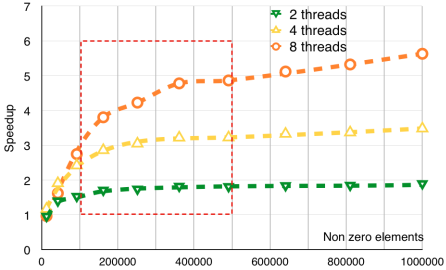



Line parallelization



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# Conclusions

- Provide a benchmark of adjustable size and more or less coupled in order to test an ODE solver
  - 1D heat diffusion model
  - Adjustable size
  - Add coupling by performing rotations
- Show the ability of this benchmark to compare two different parallelization strategies for ODE solvers
  - Block parallelization suitable for homogeneous matrices
  - Line parallelization suitable for heterogeneous matrices